King Fahd University of Petroleum & Minerals
College of Computer Sciences & Engineering
Department of Information and Computer Science

# ICS 253: Discrete Structures I

Sections 01 and 02 – Dr. Husni Al-Muhtaseb

## Final Exam – 152

**100 Minutes**

Calculators, mobile phones and other electronic devices are not permitted.

| Question | Max | Earned | CLO* | Question | Max | Earned | CLO* |
|---|---|---|---|---|---|---|---|
| 1 | 4 | | 1 | 12 | 6 | | 3 |
| 2 | 4 | | 1 | 13 | 5 | | 3 |
| 3 | 3 | | 1 | 14 | 4 | | 3 |
| 4 | 6 | | 2 | 15 | 4 | | 3 |
| 5 | 4 | | 2 | 16 | 6 | | 3 |
| 6 | 8 | | 2 | 17 | 3 | | 3 |
| 7 | 8 | | 2 | 18 | 3 | | 3 |
| 8 | 4 | | 3 | 19 | 3 | | 3 |
| 9 | 3 | | 3 | 20 | 6 | | 3 |
| 10 | 3 | | 3 | 21 | 3 | | 1 |
| 11 | 4 | | 3 | 22 | 6 | | 1 |

* **CLO** Course Learning Outcomes

| Total Out 100 | |
|---|---|

**Thursday, May 12, 2016**

## Sample Solution

**Question 1: [4 Points] Logic and Proofs**

Show that $p \to (q \to r)$ and $(p \land q) \to r$ are logically equivalent by developing a series of logical equivalences.

$$
\begin{aligned}
p \to (q \to r) &\equiv \neg p \lor (q \to r) \\
&\equiv \neg p \lor (\neg q \lor r) \\
&\equiv (\neg p \lor \neg q) \lor r \\
&\equiv \neg(p \land q) \lor r \\
&\equiv (p \land q) \to r
\end{aligned}
$$

**Question 2: [4 Points] Logic and Proofs**

Determine whether the compound proposition $(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p)$ is satisfiable.

**$(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p)$ is true when the three variable p, q, and r have the same truth value. Hence, it is satisfiable as there is at least one assignment of truth values for p, q, and r that makes it true.**

**Question 3: [3 Points] Logic and Proofs**

What is the contrapositive statement of the statement,
"If a polygon is regular, then its sides are congruent."

**If the sides of a polygon are not congruent, then it is not regular.**

**Question 4: [6 Points] Functions**

Consider the sets $A = \{0, 1, 2, 3, 4\}$ and $B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let $f$ be a function from $A$ to $B$, whose graph is $G_f = \{(0, 0),(1, 2),(2, 4),(3, 6),(4, 8)\}$

(a) Give the range of $f$.

**{0, 2, 4, 6, 8}**

(b) If the inverse, $f_{-1}$ , exists, give it as a set of ordered pairs. If it does not exist, say why not.

**The inverse does not exist because the function f is not onto.**

(c) Let $g(x) = \left\lfloor \frac{x}{2} \right\rfloor$ be a function from $B$ to $A$.

i) Give the domain of the composition function $g \circ f$.

**{0, 1, 2, 3, 4}**

ii) Give the graph of $g \circ f$ as a set of ordered pairs.

**{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4)}**

**Question 5: [4 Points] Sequences and Summations**

Describe the sequences 0, 0, 0, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 5, ... by an explicit formula.

$$g_n = \left\lfloor \frac{n}{4} \right\rfloor$$

**Question 6: [8 Points] Induction**

Prove by induction that $\sum_{k=1}^{n}(2k-1) = n^2$ for all positive integers $n$.

- Don't start by assuming $P(k + 1)$
- Two good methods:
  - Start with $P(k)$ and derive $P(k + 1)$
  - Start with LHS of goal and use $P(k)$ to show it is equal to RHS of goal

**Proof: STEP 1: For n=1 $\sum_{k=1}^{n}(2k-1) = n^2$ is true, since $1 = 1^2$ .**

**STEP 2: Suppose $\sum_{k=1}^{n}(2k-1) = n^2$ is true for some n = k ≥ 1,**
**that is $1 + 3 + 5 + . . . + (2k – 1) = k^2$.**

**STEP 3: Prove that $\sum_{k=1}^{n}(2k-1) = n^2$ is true for n = k + 1,**
**that is $1 + 3 + 5 + . . . + (2k – 1) + (2k + 1)$ ?= $(k + 1)^2$.**

**We have: $1 + 3 + 5 + . . . + (2k – 1) + (2k + 1) = k^2 + (2k + 1) = (k + 1)^2$.**

**Question 7: [8 Points] Induction and Recursion**

We are going to use structural induction to show that if $T$ is a full binary tree, then $l(T) = i(T) + 1$. Where $l(T)$ is the number of leaves of $T$ and $i(T)$ is the number of internal vertices of $T$. Note: Leaves are nodes with no children.

Below is the start of the proof. Complete the remaining parts.

Base Step: ($T$ = leaf). We have $l$(leaf) = 1 = 1 + 0 = 1 + $i$(leaf)
Induction step: Assume $l(T_1) = 1 + i(T_1)$ and $l(T_2) = 1 + i(T_2)$, where $T_1$ and $T_2$ are full binary trees and ($T = T_1 \cdot T_2$).

**We have**

| | |
|---|---|
| **$l(T_1 \cdot T_2) = l(T_1) + l(T_2)$** | **by the def. of $l$** |
| **$= 1 + i(T_1) + l(T_2)$** | **by the IH for $T_1$** |
| **$= 1 + i(T_1) + 1 + i(T_2)$** | **by the IH for $T_2$** |
| **$= 1 + i(T_1 \cdot T_2)$** | **by the def. of $i$** |

**We have completed all the cases for structural induction on a binary tree, so we can therefore conclude that for any full binary tree $T$, $l(T) = 1 + i(T)$.**

**Question 8: [4 Points] Counting and the Pigeonhole Principle**

Show that if 11 integers are selected from among {1, 2, . . . , 20}, then the selection includes integer $a$ and $b$ such that $a - b = 2$. Use Pigeonhole Principle.

**Let the pigeons be the 11 integers selected. Define 10 pigeonholes corresponding to the sets that have two integers differ by two: {3, 1}, {4, 2}, {7, 5}, {8, 6}, {11, 9}, {12, 10}, {15, 13}, {16, 14}, {19, 17}, {20, 18}. Place each integer selected into the pigeonhole corresponding to the set that contains it. Since 11 integers are selected and placed into 10 pigeonholes, some pigeonhole contains two pigeons.**

**Question 9: [3 Points] Permutations and Combinations**

A company has 12 job applicants from which to select 4 programmers and 3 management trainees. How many ways can they do this if only 7 of the applicants are qualified to be programmer? Assume no one is offered more than one job and any of the applicants can be management trainees?

$$\binom{7}{4}\binom{8}{3} = \frac{7!}{4!\,3!} \cdot \frac{8!}{3!\,5!} = 35\,.\,56 = 1960$$

**Question 10: [3 Points] Permutations and Combinations**

When we randomly select a permutation of the 26 lowercase letters of the English alphabet, what is the probability of having the first 9 letters of the permutation in alphabetical order?

**First letter can be any letter from the first 18 (26-8) letters to have the remaining in alphabet order.**

$$\frac{18}{26} \cdot \frac{1}{25} \cdot \frac{1}{24} \cdot \frac{1}{23} \cdot \frac{1}{22} \cdot \frac{1}{21} \cdot \frac{1}{20} \cdot \frac{1}{19} \cdot \frac{1}{18} = 1.588 \times 10^{-11}$$

**Question 11: [4 Points] Probability**

The random variable $X$ on a sample space $S = \{1, 2, 4, 10, 22\}$ has the following distribution:

| ***X*** | 1 | 2 | 4 | 10 | 22 |
|---|---|---|---|---|---|
| ***P*(*X*)** | 0.3 | 0.1 | 0.2 | ? | ? |

Given that P($X$ = 10) and P($X$ = 22) are equal, what is P($X$ = 10)?

**(1 – (0.3 + 0.1 + 0.2)) / 2 = 0.2**

**Question 12: [6 Points] Probability**

A box contains three coins: two regular coins and one fake two-headed coin ($P(H) = 1$),

a. You pick a coin at random and toss it. What is the probability that it lands heads up?

**Let $C_1$ be the event that you choose a regular coin, and let $C_2$ be the event that you choose the two-headed coin. We already know that $P(H|C_1) = \frac{1}{2}$ and $P(H|C_2) = 1$**

**$P(H) = P(H|C_1)\,P(C_1) + P(H|C_2)\,P(C_2) = \frac{1}{2}.\frac{2}{3} + 1.\frac{1}{3} = \frac{2}{3}$**

b. You pick a coin at random and toss it, and get heads. What is the probability that it is the two-headed coin?

$$P(C_2|H) = \frac{P(H|C_2)P(C_2)}{P(H)} = \frac{1.\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

**Question 13: [5 Points] Probability**

For three events $A$, $B$, and $C$, we know that:

$A$ and $C$ are independent,

$B$ and $C$ are independent,

$A$ and $B$ are disjoint,

$P(A \cup C) = 2/3$, $P(B \cup C) = 3/4$, $P(A \cup B \cup C) = 11/12$.

Find $P(A)$, $P(B)$, and $P(C)$.

$P(A \cap C) = P(A)\, P(C)$ as they are independent
$P(B \cap C) = P(B)\, P(C)$ as they are independent
$P(A \cap B) = 0$ as they are disjoint
$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) = 11/12$
$P(A) + P(B) + P(C) - P(B)\, P(C) - P(A)\, P(C) = 11/12$
$P(A \cup C) = P(A) + P(C) - P(A \cap C) = 2/3 \rightarrow P(A) + P(C) - P(A)\, P(C) = 2/3$
$P(B \cup C) = P(B) + P(C) - P(B \cap C) = 3/4 \rightarrow P(B) + P(C) - P(B)\, P(C) = 3/4$
We solve for P(A), P(B), and P(C)
$P(A) = 2/11$, $P(B) = 8/11$, and $P(C) = 1/12$

**Question 14: [4 Points] Sets**

Let $A \times B = \{(1, 1), (2, 2), (3, 1), (3, 2), (1, 2), (1, 4), (2, 1), (2, 4), (3, 4)\}$. Find the power set of $B$, $\mathcal{P}(B)$.

**The set B ={1, 2, 4}. Then P(B) = {∅, {1}, {2}, {4}, {1, 2}, {1, 4}, {2, 4}, {1, 2, 4}}.**

**Question 15: [4 Points] Recursive Definition**

Give a recursive definition of the set of bit strings that have the same number of zeros and ones.

**Basis Step:**
**$\lambda \in S$ ($\lambda$ is the empty string)**
**Recursive Step: If $x \in S$ and $y \in S$, then $0x1 \in S$, $1x0 \in S$ and $xy \in S$**

**Question 16: [6 Points] Linear Recurrence Relations**

Assume that the population of the world in 2015 was 7.35 billion and is growing at the rate of 1.1% a year.

a. Set up a recurrence relation for the population of the world n years after 2015.

**$a_n = a_{n-1} + 1.1\% a_{n-1} = 1.011 * a_{n-1}$, where $a_n$ is the population of the world $n$ years after 2015, and $a_0$ is the population of the world in 2015, which is 7.35 billion ($7.35 \times 10^9$).**

b. Find an explicit formula for the population of the world n years after 2015.

**$a_n = 7.35 \times 10^9 \times (1.011)^n$**

**Question 17: [3 Points] Binomial Coefficients and Identities**

What is the coefficient of $x^4y^6$ in the expansion of $(2y^3 + 5x^2)^4$?

**The coefficient of $(2y^3)^k(5x^2)^{4-k}$ in the expansion of $(2y^3 + 5x^2)^4$ is $\binom{4}{k}$, for any $0 \le k \le 4$. We are looking specifically for the coefficient of $x^4y^6$, and the term in the expansion corresponding to $k = 2$. The term in the expansion looks like**

$$\binom{4}{2}(2y^3)^2(5x^2)^2 = \left(\frac{4!}{2!\,2!} \bullet 4 \bullet 25\right)x^4y^6 = 600x^4y^6$$

**So the coefficient of $x^4y^6$ is 600.**

**Question 18: [3 Points] Binomial Coefficients and Identities**

The row of Pascal's triangle containing the binomial coefficients $\binom{10}{k}$, $0 \le k \le 10$, is:

1 10 45 120 210 252 210 120 45 10 1

Use Pascal's identity to produce the row immediately preceding this row in Pascal's triangle.

**1 9 36 84 126 126 84 36 9 1**

**Question 19: [3 Points] Linear Recurrence Relations**

Determine which of these are **L**inear **H**omogeneous **R**ecurrence **R**elations **W**ith **C**onstant **C**oefficients (**LHRRWCC**). Also, find the degree of those that are.

1. $a_n = 3a_{n-2}$

Select: [Yes, it is **LHRRWCC**] No It is not **LHRRWCC** **Degree** (if Yes): 2

2. $a_n = 3$

Select: Yes, it is **LHRRWCC** [No It is not **LHRRWCC**] **Degree** (if Yes): _____

3. $a_n = a^2_{n-1}$

Select: Yes, it is **LHRRWCC** [No It is not **LHRRWCC**] **Degree** (if Yes): _____

4. $a_n = a_{n-1} + 2a_{n-3}$

[Select: Yes, it is LHRRWCC] No It is not LHRRWCC Degree (if Yes): 3

**Question 20: [6 Points] Linear Recurrence Relations**

Solve the recurrence relation $a_n = -6a_{n-1} - 9a_{n-2}$ for $n \geq 2$ given the initial conditions $a_0 = 1$ and $a_1 = 6$.

**$c_1 = 6$, $c_2 = 9$,**

**$r^2 + 6r + 9 = 0$**

**=> $r_1 = -3$ $r_2 = -3$**

**=> $a_n = \alpha_1 (-3)^n + \alpha_2 n (-3)^n$**

**with the initial condition: $a_0 = 3$, $a_1 = -3$**

**=> $1 = \alpha_1 * (-3)^0 + \alpha_2 * 0(-3)^0 = \alpha_1$**

**$6 = -3\alpha_1 -3\alpha_2$**

**=> $\alpha_2 = -3$**

**=> $a_n = (-3)^n -3 (-3)^n n$**

**$= (-3)^n (1 - (-3)^n n)$**

**Question 21: [3 Points] Logic and Proofs**

Circle the correct answer. The statement form $(p \leftrightarrow r) \rightarrow (q \leftrightarrow r)$ is equivalent to

(a) $[(\neg p \vee r) \wedge (p \vee \neg r)] \vee \neg[(\neg q \vee r) \wedge (q \vee \neg r)]$

(b) $\neg[(\neg p \vee r) \wedge (p \vee \neg r)] \wedge [(\neg q \vee r) \wedge (q \vee \neg r)]$

(c) $[(\neg p \vee r) \wedge (p \vee \neg r)] \wedge [(\neg q \vee r) \wedge (q \vee \neg r)]$

(d) $\neg[(\neg p \vee r) \wedge (p \vee \neg r)] \vee [(\neg q \vee r) \wedge (q \vee \neg r)]$

**Question 22: [6 Points] Logic and Proofs**

You are given the following predicate on the set *P* of all people who ever lived:

Parent(*x*, *y*): true if and only if *x* is the parent of *y*.

a. Rewrite in the language of mathematical logic: All people have two parents.

**Person(x): x is a person.**

**$\forall$ x Person(x) $\rightarrow$ ($\exists$ y z Person(y) $\wedge$ Person(z) $\wedge$ parent(y, x) $\wedge$ parent(z, x) $\wedge$ y $\neq$ z)**

b. We will recursively define the concept of ancestor:
An ancestor of a person is one of the person's parents or the ancestor of (at least) one of the person's parents.
Rewrite this definition using the language of mathematical logic. Specifically, you need to provide a necessary and sufficient condition for the predicate Ancestor(*x*, *y*) to be true. (Note that you can inductively use the Ancestor(·, ·) predicate in the condition itself.)

**$\forall$ x y Person(y) $\rightarrow$ (ancestor(x, y) $\leftrightarrow$ (parent(x,y) $\vee$ ($\exists$ z parent(z,y) $\wedge$ ancestor(x,z))))**